Theorem: (Weierstrass Approximation)

Let f be a continuous, real-valued function on $[a,b]$.

Then $\exists$ a sequence of polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n \to f$ uniformly on $[a,b]$.

Proof: We assumed

$$f(0) = f(1) = 0$$

and $a = 0$, $b = 1$.

We defined

$$Q_n(t) = c_n (1 - t^2)^n$$

where $c_n = \dfrac{1}{\int_{-1}^{1} (1-t^2)^n \, dt}$

$$P_n(x) = \int_{-1}^{1} f(x+t)\,Q_n(t)\,dt$$

we showed P_n is a polynomial $\forall\, n \in \mathbb{N}$.

We want to show

$P_n \to f$ uniformly on $[0,1]$.

Step 1: Bounding c_n.

$$\frac{1}{c_n} = \int_{-1}^{1} (1-t^2)^n \, dt$$

$$= 2 \int_{0}^{1} (1-t^2)^n \, dt$$

(Since $(1-t^2)^n$ is even)

$$= 2 \int_{0}^{\frac{1}{\sqrt{n}}} (1-t^2)^n \, dt$$

$$+ 2 \int_{1/\sqrt{n}}^{1} (1-t^2)^n \, dt$$

But $1-t^2 \geq 0 \quad \forall\, t \in [0,1]$,

so $\displaystyle\int_{1/\sqrt{n}}^{1} (1-t^2)^n\,dt \geq 0$,

which implies

$$\frac{1}{c_n} \geq 2\int_0^{1/\sqrt{n}} (1-t^2)^n\,dt$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-nt^2)\,dt$$

by lemma from last class

$(1-t^2)^n \geq (1-nt^2)$

Then

$$2\int_0^{\frac{1}{\sqrt{n}}} (1 - nt^2)\,dt$$

$$= \left(2t - \frac{2nt^3}{3}\right)\Bigg|_0^{\frac{1}{\sqrt{n}}}$$

$$= \frac{2}{\sqrt{n}} - \frac{2}{3\sqrt{n}}$$

$$= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$$

Then

$$\frac{1}{c_n} > \frac{1}{\sqrt{n}}$$

$$\Rightarrow c_n < \sqrt{n}.$$

Step 2: $Q_n \to 0$ uniformly

when $|x| > \delta > 0$.

$(\delta \leq 1)$

For any such δ and for all such x,

$$
\begin{aligned}
Q_n(x) &= c_n (1-x^2)^n \\
&\leq \sqrt{n}\,(1-x^2)^n \\
&\leq \sqrt{n}\,(1-\delta^2)^n
\end{aligned}
$$

If $\delta = 1$, we get zero.

If $0 < \delta < 1$, then

$1 - \delta^2 < 1$, so

$$\boxed{\lim_{n \to \infty} \sqrt{n}\,(1-\delta^2)^n = 0},$$

independent of x.

Step 3: Finish it!

Since f is continuous on $[0,1]$, it is bounded on $[0,1]$. Let $M = \max_{x\in[0,1]} |f(x)|$.

Since $[0,1]$ is compact, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{3}$ when $|x-y| < \delta$ $(x, y \in [0,1])$

Let $\varepsilon > 0$, let δ come from the uniform continuity. Choose $x \in [0,1]$.

$$|f(x) - P_n(x)|$$

$$= \left| f(x) - \overbrace{\int_{-1}^{1} f(x+t) Q_n(t)\,dt}^{P_n(x)} \right|$$

$$= \left| f(x) \underbrace{\int_{-1}^{1} Q_n(t)\,dt}_{=1} - \int_{-1}^{1} f(x+t) Q_n(t)\,dt \right|$$

$$= \left| \int_{-1}^{1} f(x) Q_n(t)\,dt - \int_{-1}^{1} f(x+t) Q_n(t)\,dt \right|$$

$$= \left| \int_{-1}^{1} \big(f(x) - f(x+t)\big) Q_n(t)\,dt \right|$$

$$\leq \int_{-1}^{1} |f(x) - f(x+t)|\,|Q_n(t)|\,dt$$

$$= \int_{-1}^{-\delta} |f(x)-f(x+t)|\,|Q_n(t)|\,dt$$

$$+ \int_{-\delta}^{\delta} |f(x)-f(x+t)|\,|Q_n(t)|\,dt$$

$$+ \int_{\delta}^{1} |f(x)-f(x+t)|\,|Q_n(t)|\,dt$$

Middle Integral:

$$|(x+t)-x| = |t| < \delta$$

(we ignore the endpoints)

Then by uniform continuity,

$$|f(x)-f(x+t)| < \frac{\varepsilon}{3},$$ so

$$\int_{-\delta}^{\delta} |f(x)-f(x+t)||Q_n(t)|\,dt$$

$$< \frac{\varepsilon}{3} \int_{-\delta}^{\delta} |Q_n(t)|\,dt$$

$$\leq \frac{\varepsilon}{3} \int_{-1}^{1} |Q_n(t)|\,dt$$

But $Q_n(t) = c_n(1-t^2)^n$

≥ 0 on $[-1,1]$,

so

$$\int_{-1}^{1} |Q_n(t)|\,dt = \int_{-1}^{1} Q_n(t)\,dt$$

$$= 1$$

Hence, the middle integral

is less than $\frac{\varepsilon}{3}$.

Upper + Lower Integrals

Concentrate on

$$2\int_{\delta}^{1} |f(x)-f(x+t)|\,|Q_n(t)|\,dt$$

$$= 2\int_{\delta}^{1} |f(x)-f(x+t)|\,\underbrace{Q_n(t)}_{Q_n \geq 0}\,dt$$

$$\leq 2\int_{\delta}^{1} \underbrace{\left(|f(x)|+|f(x+t)|\right)}_{\text{triangle inequality}}Q_n(t)\,dt$$

$$\leq 2 \int_{\delta}^{1} (M+M) \, Q_n(t)\, dt$$

$$= 4M \int_{\delta}^{1} Q_n(t)\, dt$$

We know $Q_n \to 0$ uniformly on $|t| > \delta$. Choose N so that $Q_n(t) < \dfrac{\varepsilon}{12M(1-\delta)}$.

Then

$$4M \int_{\delta}^{1} Q_n(t)\,dt$$

$$< 4M \int_{\delta}^{1} \frac{\varepsilon}{24M(1-\delta)}\,dt$$

$$= \frac{\varepsilon}{3\cdot(1-\delta)} \cdot (1-\delta) = \frac{\varepsilon}{3}.$$

Hence, the lower integral is bounded by $\varepsilon/6$.

But the same calculation applies to the upper integral, so we get

$$|f(x) - P_n(x)|$$

$$< (\text{upper}) + (\text{middle}) + (\text{lower})$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon \qquad \square$$

(modulo assumptions)